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RELAXATION PHENOMENA IN SATURATED POROUS MEDIA. LINEAR THEORY*

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Isothermal relaxation processes in a non-deformable porous specimen of finite size saturated with a weakly compressible liquid are considered. experiments with the establishment of stationary seepage of the liquid show (see Sect.2) that the non-stationary seepage equations describing the elastic flow model /1, 2/ do not apply for porous specimens with characteristic dimension of the order of several meters. Non-stationary processes in these specimens naturally should be described using more-general relaxation models of seepage /3, 4/. We investigate the general properties of the relaxation kernel characterizing the given "liquid-porous medium" system. A solution of the problem of establishment of stationary seepage is given. The theory developed in this paper is applied to analyse experimental results on non-stationary seepage of low-viscosity weakly compressible liquids. This leads to some general conclusions regarding the form of relaxation kernels for particular systems.

1. Consider an isotropic homogeneous non-deformable porous medium that fills a bounded region D in the Euclidean space R^n $(n = 1, 2, 3)$. The values $n = 1$ and $n = 2$ correspond to one- and two-dimensional seepage problems. We assume that the boundary of the region ∂D is a C^1-submanifold in R^n.

The porous medium is saturated with a liquid. We will investigate processes in which the liquid density ρ is nearly equal to a constant ρ_0, and we may accordingly use a linear expression for the pressure (E is the bulk modulus of elasticity of the liquid)

$$p = p_0 + E(\rho - \rho_0)/\rho_0 \tag{1.1}$$

In relaxation seepage theory /3, 4/, D'Arcy's law is generalized in the form

$$\mathbf{u}(t_0, \mathbf{r}) = -k\mu^{-1} \int K(t_0 - t) \nabla G(t, \mathbf{r})dt, \quad G = p + \varphi\rho \tag{1.2}$$

Here $\mathbf{u}$ is the seepage velocity, k the permeability, φ the gravitational potential, and μ the viscosity of the liquid, which may be assumed constant. All integrals are between the limits from $-\infty$ to $+\infty$, unless otherwise specified. The kernel $K = K(t)$ is independent of the spatial coordinates and characterizes the internal relaxation processes in the "porous medium-liquid" system. The function $K = K(t)$ satisfies a number of conditions that follows from physical and thermodynamic considerations. Let us list these conditions.

If ∇G varies over time, preserving a constant direction at a given point in space, we naturally assume that the corresponding seepage velocity points in the opposite direction at the given point at all instants of time. This assumption is equivalent to the following condition:

1°. $K = K(t)$ is a non-negative function (possibly generalized) with dimensions of (time)$^{-1}$.

For time-constant ∇G, the relationship (1.2) should reduce to D'Arcy's law, whence

2°. $\int K(t)\,dt = 1$.

The kernel $K = K(t)$ describes the effect of the field ∇G on the seepage velocity $\mathbf{u}$. By the casuality principle, the field $\nabla G(t, \mathbf{r})$ cannot affect $\mathbf{u}(t_0, \mathbf{r})$ for $t > t_0$. The values of $\nabla G(t, \mathbf{r})$ for $t = t_0$ may affect $\mathbf{u}(t_0, \mathbf{r})$ in a special, singular manner. We accordingly have the following condition.

3°. The support of the function $K = K(t)$ lies on the halfline $[0, +\infty)$. The function $K = K(t)$ may have a singular support - the point $t = 0$. For example, D'Arcy's law is a special case of the law (1.2) for a kernel equal to the Dirac δ-function.

Since the effect of $\nabla G(t, \mathbf{r})$ on $\mathbf{u}(t_0, \mathbf{r})$ should decrease as the difference $(t_0 - t)$ increases, we naturally assume that

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4°. For large t, the function $K = K(t)$ is smooth and rapidly decreasing.
Now, consider the quantity

$$A = -\int \mathbf{u}(t, \mathbf{r}) \nabla G(t, \mathbf{r})\, dt \tag{1.3}$$

for an arbitrary vector function $\nabla G(t, \mathbf{r})$ rapidly decreasing as $|t| \to +\infty$. The quantity A is proportional to the work of the friction forces of the liquid with the skeleton of the rock in a porous particle. By the second law of thermodynamics, A is always non-negative. Hence it follows that

5°. For any function $f = f(t)$ that decreases rapidly as $|t| \to +\infty$, we have the inequality

$$\int dt_1 \int dt_2 K(t_1 - t_2) f(t_1) f(t_2) \geqslant 0$$

For non-zero functions $f = f(t)$, the inequality should be strict.

Conditions 1°-5° lead to a number of corollaries regarding the Fourier transform of the kernel $K = K(t)$,

$$K_F(\omega) = \int e^{-i\omega t} K(t)\, dt, \quad \omega \in R$$

By condition 2°, $K_F(0) = 1$. Now, since $K = K(t)$ is real valued, we have

$$\overline{K_F(\omega)} = K_F(-\omega), \quad \omega \in R \tag{1.4}$$

By condition 1° and Bochner's theorem /5/ it follows that $K_F = K_F(\omega)$ is a positive definite function. This means that for any complex numbers z_j and any real numbers ω_j $(j = 1, \ldots, N)$, we have the inequality

$$\sum_{i, j=1}^{N} \bar{z}_i z_j K_F(\omega_i - \omega_j) \geqslant 0 \tag{1.5}$$

Let $n = 2, \omega_1 = \omega, \omega_2 = 0$. From (1.4) and (1.5) it follows that for any complex numbers z_1 and z_2 we have the inequality

$$(|z_1|^2 + |z_2|^2) K_F(0) + 2\mathrm{Re}\,(\bar{z}_1 z_2 K_F(\omega)) \geqslant 0$$

which is equivalent to the inequality

$$|K_F(\omega)| \leqslant 1, \quad \omega \in R \tag{1.6}$$

From 3° it follows that the function $K_F = K_F(\omega)$ has an analytic continuation in the lower half-plane of the complex plane ω /6/. Relationship (1.4) is accordingly continued to the complex plane

$$\overline{K_F(\bar{\omega})} = K_F(-\omega), \quad \omega \in C \tag{1.7}$$

In terms of Fourier transforms, condition 5° combined with (1.4) is equivalent to the inequality

$$\int_0^\infty |f_F(\omega)|^2 \,\mathrm{Re}\, K_F(\omega)\, d\omega \geqslant 0$$

Since $f_F(\omega)$ is arbitrary we thus obtain that $\mathrm{Re} K_F(\omega)$ is non-negative for $\omega \geqslant 0$. In what follows, we will use a stronger inequality

$$\mathrm{Re}\, K_F(\omega) > 0, \quad \omega \in R \tag{1.8}$$

which simplifies the analysis and is apparently always true in applications.

By conditions 3° and 4°, the function $K_F(\omega)$ tends to the same constant for $\omega \to \pm\infty$. By this remark and inequality (1.8), it follows from general theory /7/ that the holomorphic function $K_F = K_F(\omega)$ has no zeros for $\mathrm{Im}\,\omega < 0$. Thus, the complex function $K_F = K_F(\omega)$ is a conformal mapping of the half-plane $\mathrm{Im}\,\omega < 0$ on some region in the circle $|z| < 1$, $\mathrm{Re}\, z > 0$, $z \in C$.

In what follows we assume that the function $K_F = K_F(\omega)$ is analytically continuable to the upper half-plane of the complex plane and is a meromorphic function /7/. There are no poles if $K_F(\omega) = 1$, i.e., when there are no internal relaxation processes.

Let S_1, S_2 be non-empty open subsets in ∂D, such that the boundaries of S_1 and S_2 in ∂D coincide and constitute a C^1-submanifold of the manifold ∂D.

Consider the following auxiliary problem of determining the function $f \in W_2^1(D)$:

$$\Delta_n f = 0;\ f\,|_{S_1} = f_1,\ \partial f/\partial n\,|_{S_2} = f_2 \qquad (1.9)$$
$$f_1 \in W_2^{1/2}(S_1),\ f_2 \in W_2^{-1/2}(S_2)$$

Here Δ_n is the n-dimensional Laplace operator; the notation of functional spaces is from /8/.

Problem (1.9) has a unique solution /8/, which is representable in the form $f = L_1 f_1 + L_2 f_2$, where L_1 and L_2 are appropriate linear operators.

Let us now investigate the problem of establishing stationary seepage in the following setting. Assume that for $t < 0$ the liquid in the porous medium is at rest, $p\,|_{t<0} = p_0$, and for $t \geqslant 0$ the pressure and flow distributions are specified discontinuously in time on the boundary of the porous specimen (n is the inner normal to ∂D) in the form

$$p\,|_{S_1} = g_1,\ (\mathbf{u}, n)\,|_{S_2} = g_2,\ t \geqslant 0 \qquad (1.10)$$
$$g_1 \in W_2^{1/2}(S_1),\ g_2 \in W_2^{-1/2}(S_2)$$

The dynamics of pressure variation for $t \geqslant 0$ in the porous specimen is determined by the equation of continuity

$$\partial(m\rho)/\partial t + \nabla(\rho\mathbf{u}) = 0 \qquad (1.11)$$

where m is the porosity, and also be relationships (1.1) and (1.2).

Gravitational effects are ignored. From (1.1), (1.2), and (1.11), we obtain the equation

$$\frac{\partial p}{\partial t}(t_0, \mathbf{r}) = \varkappa \int K(t_0 - t)\,\Delta_n p(t, \mathbf{r})\,dt, \quad \varkappa = \frac{kE}{\mu m} \qquad (1.12)$$

The boundary conditions (1.10) are transformed to

$$p|_{S_1} = g_1, \qquad \int\limits_{-\infty}^{t} K(t - t_0)\frac{\partial p}{\partial n}(t_0, \mathbf{r})\,|_{S_2}\,dt_0 = -\frac{\mu}{k}g_2, \quad t \geqslant 0 \qquad (1.13)$$

Fourier-Laplace transforming by t the equations (1.12) and (1.13) /6/, we obtain

$$(i\omega - \varkappa K_F(\omega)\,\Delta_n)\,p_F = p_0$$
$$p_F\,|_{S_1} = -g_1\frac{i}{\omega}, \quad \left.\frac{\partial p_F}{\partial n}\right|_{S_2} = \frac{\mu}{k}g_2\frac{i}{\omega K_F(\omega)} \qquad (1.14)$$
$$p_F = \int\limits_0^{+\infty} e^{-i\omega t}p(t)\,dt$$

We introduce a new unknown function $G = G(\omega, \mathbf{r})$, defined by

$$p_F = G + H, \quad H = -\frac{i}{\omega}L_1 g_1 + \frac{i\mu}{k\omega K_F(\omega)}L_2 g_2 \qquad (1.15)$$

Then from (1.14) we obtain

$$(i\omega - \varkappa K_F(\omega)\Delta_n)\,G = p_0 + i\omega H \qquad (1.16)$$
$$G\,|_{S_1} = 0,\ \partial G/\partial n\,|_{S_2} = 0 \qquad (1.17)$$

The pressure $p = p(t, \mathbf{r})$ can be recovered from $p_F = p_F(\omega, \mathbf{r})$ using the formula

$$p(t, \mathbf{r}) = (2\pi)^{-1}\int e^{i\omega t + \varepsilon t}p_F(\omega - i\varepsilon, \mathbf{r})\,d\omega \qquad (1.18)$$

where ε is an arbitrary positive quantity. The integral in (1.18) can be evaluated using the residue formula

$$p(t, \mathbf{r}) = i\sum_j \operatorname{Res}_{\omega_j}[e^{i\omega t}p_F(\omega, \mathbf{r})] \qquad (1.19)$$

where ω_j are the poles of the function $p_F(\omega, \mathbf{r})$. From (1.14)-(1.16) we see that the poles ω_j are divided into three subsets: a) the single point $\omega_0 = 0$, b) ω_{1j} - the roots of the equation $K_F(\omega) = 0$, and c) ω_{2j} - the values of ω for which the operator $(i\omega - \varkappa K_F(\omega)\,\Delta_n)$

does not have an inverse. In the latter case, Δ_n is a selfadjoint operator in $L_2(D)$ defined by the boundary conditions (1.17) /8/.

The term corresponding to ω_0 in (1.19) can be separated in explicit form. Then

$$p(t, \mathbf{r}) = L_1 g_1 - \mu k^{-1} L_2 g_2 + R(t, \mathbf{r})$$

$$R(t, \mathbf{r}) = i \sum_j \mathrm{Res}_{\omega_{1j}} [e^{i\omega t} p_F(\omega, \mathbf{r})] + i \sum_j \mathrm{Res}_{\omega_{2j}} [e^{i\omega t} p_F(\omega, \mathbf{r})]$$

We see that the complex numbers ω_{1j}, ω_{2j} have positive imaginary parts, and therefore the term $R(t, \mathbf{r})$ describes relaxation of the system to steady-state seepage.

Indeed, the fact that the poles ω_{1j} lie in the upper half-plane of the complex plane follows from the previous assumptions regarding the function $K_F = K_F(\omega)$. On the other hand, let ω_{2j} be a number that satisfies one of the equations

$$i\omega + \varkappa K_F(\omega) \lambda_l = 0, \quad 0 < \lambda_l < \lambda_{l+1} \; (l = 0, 1, 2, \ldots) \tag{1.20}$$

where λ_l is the spectrum of the operator $(-\Delta_n)$. If $\mathrm{Im}\, \omega_{2j} \leqslant 0$, then from (1.20) we obtain the inequality

$$\mathrm{Re}\, K_F(\omega_{2j}) = (\varkappa\lambda_l)^{-1} \mathrm{Im}\, \omega_{2j} \leqslant 0$$

which contradicts (1.8).

Assume that the relaxation to steady-state seepage is observed experimentally. Then, from the complete pressure field $p(t, \mathbf{r})$ we can separate the term $R(t, \mathbf{r})$, and applying Fourier analysis to the function $R(t, \mathbf{r})$ separate the complex frequencies ω_{1j} and ω_{2j}. These frequencies correspond to a denumerable set of relaxation processes in the porous medium, and the quantities $\tau_{aj} = (\mathrm{Im}\, \omega_{aj})^{-1}$ are the corresponding relaxation times. (Note that by (1.7), (1.20), the sets $\Omega_1 = \{\omega_{1j}\}$, $\Omega_2 = \{\omega_{2j}\}$ are invariant under the transformation $\omega \mapsto (-\overline{\omega})$).

In an experiment, it is of course easiest to separate the process with the largest τ_{aj} (the smallest $\mathrm{Im}\, \omega_{aj}$), because the modes with lower relaxation times "die out" faster. We will show that at least for large specimens the main relaxation process has a pure imaginary complex frequency, which is given by the equation

$$i\omega + \varkappa K_F(\omega) \lambda_0 = 0 \tag{1.21}$$

Consider specimens of the porous medium which have identical shape but different characteristic dimensions L. We see that λ_l depends on L in a fairly simple manner, $\lambda_l(L) = \lambda_l(L_0)(L_0/L)^2$. In Eq.(1.21) set $\omega = iy$ and $F(y) = K_F(iy)$. Then (1.21) takes the form

$$\varkappa\lambda_0 = y/F(y) \tag{1.22}$$

Since the right-hand side of Eq.(1.22) vanishes for $y = 0$ and increases monotonically for small positive y, Eq.(1.22) has a real solution at least for sufficiently large L with the asymptotic expression

$$y_0 = \varkappa\lambda_0(L) + O(L^{-4}) = \varkappa\lambda_0(L_0)(L_0/L)^2 + O(L^{-4})$$

The corresponding relaxation time

$$\tau_r = 1/y_0 \tag{1.23}$$

for sufficiently large L is greater than the relaxation times produced by other solutions of Eq.(1.22) or by solutions of Eq.(1.20) for $l \neq 0$. Since the relaxation times produced by the equation $K_F(\omega) = 0$ are independent of L, then for large specimens formula (1.23) gives the leading relaxation time.

Let $\tau' = (\varkappa\lambda_0)^{-1}$. We see from (1.22) that τ' is identical with τ_r when $F \equiv 1$, i.e., when there are no internal relaxation processes in the "porous medium-liquid" system. Therefore the difference $\tau = \tau_r - \tau'$ characterizes the deviation of seepage from elastic flow. For large specimens,

$$\tau = -\int_0^\infty tK(t)\, dt + O(L^{-2}) \tag{1.24}$$

Thus observations of the establishment of the stationary mode in principle can be used to obtain the required information for the recovery of the relaxation kernel $K = K(t)$. For example, if we observe the leading relaxation process on specimens of different size, then from (1.22) we can obtain the function $F(y)$ and then apply analytic continuation to obtain the function $K_F(\omega)$ and hence the kernel $K(t)$. In a more special case, when we know the general functional form of the kernel $K = K(t)$ with one arbitrary parameter, this parameter may be estimated by experimental measurement of the parameter τ.

2. In order to determine the internal relaxation characteristics of the "porous medium-liquid" system, we made observations of the establishment of stationary seepage of acetone and toluene in a cylindrical specimen formed by dense packing of ground quartz sand. The length of the specimen was $L = 5$ m, its diameter $d = 2.6\mathrm{x}10^{-2}$ m, porosity $m = 0.222$. Before filling with liquid, all the components of the measuring system and the porous specimen were placed in a vacuum chamber, and the liquid was initially degassed. In the notation of Sect.1, we have $n = 1$, $D = [0, L]$, $S_1 = \{0\}$, $S_2 = \{L\}$. The pressure in S_1 and the flow in S_2 were given, and we measured the pressure $p(t)$ in S_2, which exponentially approached a constant level p_e.

The experimental dependences were analysed by Padé-approximation of the Laplace transform of the function $p(t)$ /9/. The analysis produced the leading relaxation time τ_r. Once the system has stabilized in the stationary mode, we determined the permeability k. Then, using the spectral expression $\lambda_l = [\pi(2l+1)/(2L)]^2$ $(l = 0, 1, \ldots)$, we calculated τ' and τ. The experiments were conducted for various volumetric flow rates Q. The initial pressure in all experiments was set at $p_0 = 12.4$ MPa. The results are summarized in Table 1 (the first three lines for toluene, and the last three lines for acetone).

$Q \cdot 10^{8}$, m^3/sec	p_e, MPa	$k \cdot 10^{13}$, m^2	τ_r, sec	τ', sec	τ, sec
0,083	12.8	13,0	700	91	609
1,67	19,3	13,3	760	89	671
3,33	26,4	13,1	800	90	710
6,67	27,4	13,6	666	44	622
3,40	20,1	13,5	614	44	570
0,22	12,9	13,6	604	44	560

It is remarkable that in all experiments we obtained $\tau > 0$ (contrary to (1.24)). According to (1.22), the inequality $\tau > 0$ is possible only if

$$0 < F(1/\tau_r) < 1 \qquad (2.1)$$

Consider the simplest relaxation kernels proposed in /3/. Their Fourier transforms are

$$K_{1F}(\omega) = 1 + i\omega\tau_p;\ \tau_p > 0 \qquad (2.2)$$

$$K_{2F}(\omega) = (1 + i\omega\tau_u)^{-1};\ \tau_u > 0 \qquad (2.3)$$

$$K_{3F}(\omega) = (1 + i\omega\tau_p)(1 + i\omega\tau_u)^{-1};\ \tau_p,\ \tau_u > 0 \qquad (2.4)$$

The kernel (2.2) is formally consistent with (2.1), when $\tau_p \equiv \tau$. However, the kernel (2.2) is inconsistent with (1.6), i.e., with the condition of positive definiteness. The kernel (2.3) satisfies all the general conditions 1°-5° imposed on kernels, but is inconsistent with (2.1).

The kernel (2.4) satisfies the conditions 1°-5° if $\tau_p < \tau_u$. Here, $F_3(y) = K_{3F}(iy)$ also satisfies the inequality (2.1) for $y > 1/\tau_p$. However, the region $y > 1/\tau_p$ corresponds to faster relaxation processes than the relaxation process associated with the zero of the function $F_3 = F_3(y) : y_1 = 1/\tau_p$. Therefore, if we use the kernel (2.4) to interpret our experimental data, we must simply set $\tau_r = \tau_p$. If we additionally take $\tau_u = 2\tau_p$ in accordance with the results of /3/, then the parameters of the relaxation kernel are fully defined.

This interpretation of the experimental results using the kernel (2.4) is not entirely satisfactory. Indeed, it assumes that the length L of the porous specimen accidentally falls in the range where the relaxation to steady-state seepage is dominated by the internal relaxation processes in the "porous medium-liquid" system. A more natural assumption is that the function $F = F(y)$ is positive in some interval $y \in (0, a)$, $a > 0$, $F(a) = 0$, and for $y \in [0, a)$ the function $H_1(y) = y/F(y)$ is monotone increasing. Then the leading relaxation time is given by Eqs.(1.22) and (1.23).

The graphical solution of Eq.(1.22) for the chosen function $F = F(y)$ is qualitatively shown in the figure, where Curves *1* and *2* correspond to $F(y)$ and $y/F(y)$. If we approximate $F = F(y)$ near $y = a$ by the expression $F(y) = -K_1(y - a)$, $K_1 > 0$, then Eq.(1.22) can be used to find an asymptotic solution for small L,

$$\tau_r = a^{-1} + (K_1 a)^{-1}\tau' + O(L^4)$$

The available experimental results are insufficient in order to determine the parameters K_1 and a. Nevertheless, if we assume that the specimen is sufficiently small, we obain that $a \approx 1/\tau_r$.

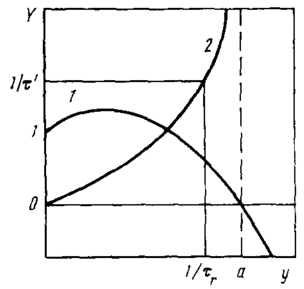

Fig.1

The theoretical and experimental studies thus indicate that non-stationary seepage of liquids should be described in the framework of relaxation seepage theory. The elastic flow equations for processes with characteristic times of the order of 10^3 give results at variance with the experimental findings. Experimental studies also indicate that the Fourier transform of the relaxation kernel should take values in (0, 1) over a certain interval of the imaginary axis. This condition considerably restricts the class of physically meaningful relaxation kernels.

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